Exam Analysis on Manifolds

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This exam consists of **three** assignments. You get 10 points for free.

Assignment 1. (30 pt.)
Let $x, y, z : \mathbb{R}^3 \to \mathbb{R}$ be the coordinate functions on $\mathbb{R}^3$. The map $\eta : \mathcal{X}(\mathbb{R}^3) \to C^\infty(\mathbb{R}^3)$ maps a vector field $X = a_1E_1 + a_2E_2 + a_3E_3$ on $\mathbb{R}^3$, where $a_i : \mathbb{R}^3 \to \mathbb{R}$ are C^∞-functions, to the function $\eta(X)$ given by

$$\eta(X) = x\, a_2 + y\, a_3 + z\, a_1.$$

Note that $E_1 = \dfrac{\partial}{\partial x}$, $E_2 = \dfrac{\partial}{\partial y}$ and $E_3 = \dfrac{\partial}{\partial z}$.

1. (8 pt.) Prove that η is a differential one-form[1], and that $\eta = z\, dx + x\, dy + y\, dz$.
2. (8 pt.) Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be the map given by $f(u, v) = (u + v, u - v, uv)$. Determine functions $a, b : \mathbb{R}^2 \to \mathbb{R}$ such that $f^*\eta = a\, du + b\, dv$.
3. (7 pt.) Compute $\omega = \eta \wedge d\eta$.
4. (7 pt.) Prove that ω is exact, and determine a differential two-form σ such that $\omega = d\sigma$.

Assignment 2. (30 pt.)
Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a C^∞ function with zero set $M = f^{-1}(0)$. Furthermore,

$$\left(\frac{\partial f}{\partial x}(p), \frac{\partial f}{\partial y}(p)\right) \neq (0, 0) \text{ for } p \in M.$$

The map $i : M \to \mathbb{R}^3$ is the inclusion map.

1. (6 pt.) Prove that M is a 2-dimensional submanifold of $\mathbb{R}^3$.
2. (6 pt.) Determine a basis of T_pM, for $p \in M$.
3. (4 pt.) Let σ be the differential one-form on $\mathbb{R}^3$ given by $\sigma = df$.
   Prove that $i^*\sigma = 0$.
4. (7 pt.) Let $\Omega = dx \wedge dy \wedge dz$ be the volume form on $\mathbb{R}^3$, and let X be the vector field on $\mathbb{R}^3$ given by
   $$X = \frac{\partial f}{\partial x} E_1 + \frac{\partial f}{\partial y} E_2 + \frac{\partial f}{\partial z} E_3.$$
   Prove that
   $$\iota_X\Omega = \frac{\partial f}{\partial z} dx \wedge dy + \frac{\partial f}{\partial x} dy \wedge dz + \frac{\partial f}{\partial y} dz \wedge dx.$$
   (Recall that $\iota_X\Omega$ is the differential two-form given by $\iota_X\Omega(Y, Z) = \Omega(X, Y, Z)$.)
5. (7 pt.) Prove that $i^*(\iota_X\Omega)$ is a nowhere zero differential two-form on M.

Assignment 3 on next page

[1] Equivalently: a differential form of degree one

Assignment 3. (30 pt.)
Note: the solution of this assignment is likely to be shorter than its statement.

Let M be a compact differentiable manifold, and let $\varphi : M \to \mathbb{S}^2$ be a C^∞-map. In part 4 you are asked to prove that every closed differential one-form on the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ is exact.

1. (7 pt.) If φ is a diffeomorphism then every closed differential one-form on M is exact. Prove this, assuming the claim in part 4 holds.

We are going to use this result to prove that the torus and the two-sphere are not diffeomorphic. Let M be the torus of revolution in $N := \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \neq (0, 0)\}$ (the complement of the z-axis in $\mathbb{R}^3$) obtained by rotating the circle in the xz-plane with radius r and center $(R, 0, 0)$ around the z-axis, where $0 < r < R$, and let $j : M \to N$ be the inclusion map. On N we consider the differential form η, given by

$$\eta = \frac{1}{x^2 + y^2} (-y\, dx + x\, dy). \tag{1}$$

Let $\omega = j^*\eta$ be its pull-back to the torus M.

2. (7 pt.) Prove that ω is a closed differential one-form on M.

3. (7 pt.) Prove that ω is not an exact differential one-form on M.

   Hint: Let $\psi : \mathbb{S}^1 = \{(u, v) \in \mathbb{R}^2 \mid u^2 + v^2 = 1\} \to M$ be the embedding given by

   $$\psi(u, v) = \big((R + r)\, u, (R + r)\, v, 0\big),$$

   and prove that $\int_{\mathbb{S}^1} \psi^*\omega \neq 0$.

4. (9 pt.) Prove that every closed differential one-form on $\mathbb{S}^2$ is exact. Conclude that the torus and the two-sphere are not diffeomorphic.

   Hint: Let ω be a closed differential one-form on $\mathbb{S}^2$. Let (f_1, U_1) and (f_2, U_2) be the charts given by (inverse) sterographic projection from the north pole $p_1 = (0, 0, 1)$ and the south-pole $p_2 = (0, 0, -1)$, respectively. So in particular, $U_1 = U_2 = \mathbb{R}^2$, and $f_1(U_1) \cap f_2(U_2)$ is connected. Use Poincaré's lemma to conclude that there are functions $\varphi_i : f_i(U_i) \to \mathbb{R}$ with $\omega|_{f_i(U_i)} = d\varphi_i$, and conclude that $\varphi_2 - \varphi_1$ is constant on the connected set $f_1(U_1) \cap f_2(U_2)$. Use this to define a function φ with $d\varphi_i = d\varphi$ on $f_i(U_i)$.

Solutions

Assignment 1.

1. It is easy to check that, for vector fields X and Y on $\mathbb{R}^3$ and for $f \in C^\infty(\mathbb{R}^3)$: $\eta(X+Y) = \eta(X) + \eta(Y)$ and $\eta(fX) = f\eta(X)$. Therefore, η is a differential one-form. Now

$$\begin{aligned} \eta &= \eta(E_1)\,dx + \eta(E_2)\,dy + \eta(E_3)\,dz \\ &= z\,dx + x\,dy + y\,dz. \end{aligned} \tag{2}$$

2. A straightforward computation shows that

$$\begin{aligned} f^*(dx) &= d(x\circ f) = d(u+v) = du + dv, \\ f^*(dy) &= d(y\circ f) = d(u-v) = du - dv, \\ f^*(dz) &= d(z\circ f) = d(uv) = v\,du + u\,dv. \end{aligned}$$

Therefore,

$$\begin{aligned} f^*\eta &= (z\circ f)\,f^*(dx) + (x\circ f)\,f^*(dy) + (y\circ f)\,f^*(dz) \\ &= uv\,(du+dv) + (u+v)(du-dv) + (u-v)(v\,du + u\,dv) \\ &= (2uv - v^2 + u + v)\,du + (u^2 - u - v)\,dv. \end{aligned}$$

In other words, $a(u,v) = 2uv - v^2 + u + v$ and $b(u,v) = u^2 - u - v$.

3. Using (2) we get $d\eta = dz\wedge dx + dx\wedge dy + dy\wedge dz$, so

$$\begin{aligned} \eta\wedge d\eta &= z\,dx\wedge dy\wedge dz + x\,dy\wedge dz\wedge dx + y\,dz\wedge dx\wedge dy \\ &= (x+y+z)\,dx\wedge dy\wedge dz. \end{aligned}$$

4. Every differential 3-form on $\mathbb{R}^3$ is closed, and, hence, exact according to Poincaré's Lemma. Therefore, there is a differential 2-form σ on $\mathbb{R}^3$ with $\omega = d\sigma$. Let $\sigma = f\,dy\wedge dz$, then $d\sigma = \frac{\partial f}{\partial x}\,dx\wedge dy\wedge dz$. Take, e.g., $f(x,y,z) = \frac{1}{2}x^2 + xy + xz$, in other words, take

$$\sigma = (\tfrac{1}{2}x^2 + xy + xz)\,dy\wedge dz.$$

Assignment 2.

1. Since $df_p : T_p\mathbb{R}^3 \to \mathbb{R}$ has rank one for all $p \in M$, the zero-set M of f is a two-dimensional submanifold of $\mathbb{R}^3$.

2. Since $T_pM = \ker df_p$, a vector $v = v_1e_1 + v_2e_2 + v_3e_3$ is a tangent vector of M at p iff

$$v_1\frac{\partial f}{\partial x}(p) + v_2\frac{\partial f}{\partial y}(p) + v_3\frac{\partial f}{\partial z}(p) = 0.$$

A non-zero solution is $v = -\frac{\partial f}{\partial y}(p)\,e_1 + \frac{\partial f}{\partial x}(p)\,e_2$. A solution w with a non-zero component in the e_3-direction is

$$w = \begin{cases} -\frac{\partial f}{\partial z}(p)\,e_1 + \frac{\partial f}{\partial x}(p)\,e_3, & \text{if } \frac{\partial f}{\partial x}(p) \neq 0, \\ -\frac{\partial f}{\partial z}(p)\,e_2 + \frac{\partial f}{\partial y}(p)\,e_3, & \text{if } \frac{\partial f}{\partial x}(p) = 0. \end{cases}$$

Since $\{v, w\}$ is an independent system (and T_pM is two-dimensional), it is a basis of T_pM.

3. Straightforward, since $i^*(df) = d(f \circ i) = 0$ (since $f(i(p)) = 0$, for $p \in M$).

4. Let $\omega = \iota_X \Omega$, then

$$\omega = \omega(E_1, E_2)\, dx \wedge dy + \omega(E_2, E_3)\, dy \wedge dz + \omega(E_3, E_1)\, dz \wedge dx.$$

Therefore, the claim follows from

$$\omega(E_1, E_2) = \Omega(\frac{\partial f}{\partial x} E_1 + \frac{\partial f}{\partial y} E_2 + \frac{\partial f}{\partial z} E_3, E_1, E_2) = \frac{\partial f}{\partial z},$$

$$\omega(E_2, E_3) = \Omega(\frac{\partial f}{\partial x} E_1 + \frac{\partial f}{\partial y} E_2 + \frac{\partial f}{\partial z} E_3, E_2, E_3) = \frac{\partial f}{\partial x},$$

$$\omega(E_3, E_1) = \Omega(\frac{\partial f}{\partial x} E_1 + \frac{\partial f}{\partial y} E_2 + \frac{\partial f}{\partial z} E_3, E_3, E_1) = \frac{\partial f}{\partial y}.$$

5. Let $p \in M$, and let $\{v, w\}$ be the basis of T_pM determined in step 2. First observe that

$$(i^*(\iota_X \Omega))_p(v, w) = (\iota_X \Omega)_{i(p)}(di_p(v), di_p(w)) = \Omega_p(X(p), v, w).$$

(Note that $di_p : T_pM \to T_p\mathbb{R}^3$ is the inclusion map, so $di_p(v) = v$ and $di_p(w) = w$.) So we have to prove that $\Omega(X(p), v, w) \neq 0$. Assume $\frac{\partial f}{\partial x}(p) \neq 0$. Then

$$\Omega(X(p), v, w) = \begin{vmatrix} \frac{\partial f}{\partial x}(p) & \frac{\partial f}{\partial y}(p) & \frac{\partial f}{\partial z}(p) \\ -\frac{\partial f}{\partial y}(p) & \frac{\partial f}{\partial x}(p) & 0 \\ -\frac{\partial f}{\partial z}(p) & 0 & \frac{\partial f}{\partial x}(p) \end{vmatrix}$$

$$= \frac{\partial f}{\partial x}(p)\big(\frac{\partial f}{\partial x}(p)^2 + \frac{\partial f}{\partial y}(p)^2 + \frac{\partial f}{\partial z}(p)^2\big) \neq 0.$$

If $\frac{\partial f}{\partial x}(p) = 0$ the argument is similar.

Assignment 3.

1. Let ω be a closed differential one-form on M, then $\sigma := (\varphi^{-1})^*\omega$ is a closed differential one-form on $\mathbb{S}^2$. Therefore, σ is exact, i.e., $\sigma = df$ for some C^∞ function $f : \mathbb{S}^2 \to \mathbb{R}$. But then $\omega = \varphi^*(df) = d(f \circ \varphi)$. In other words, ω is exact.

2. A straightforward computation shows that $d\eta = 0$, so $d\omega = d(j^*\eta) = j^*(d\eta) = 0$.

3. A straightforward computation shows that

$$\psi^*\omega = (\psi \circ j)^*\eta = \frac{1}{u^2 + v^2}\,(-v\, du + u\, dv),$$

so $\int_{\mathbb{S}^1} \psi^*\omega = 2\pi \neq 0$. If $\omega = df$, then $\int_{\mathbb{S}^1} \psi^*\omega = \int_{\mathbb{S}^1} d(f \circ \psi) = \int_{\partial \mathbb{S}^1} f \circ \psi = 0$, since $\partial \mathbb{S}^1 = \emptyset$.

4. The hint is almost the solution. Let $\omega_i = f_i^*(\omega)$, then ω_i is a closed differential

one-form on $\mathbb{R}^2$. By Poincaré's lemma, it is exact, so there are functions $\psi_i : U_i \to \mathbb{R}$ with $d\omega_i = d\psi_i$. Define $\varphi_i = \psi_i \circ f_i^{-1} : f_i(U_i) \to \mathbb{R}$, then $d(\varphi_1 - \varphi_2) = 0$ on $W := f_1(U_1) \cap f_2(U_2)$. Since W is connected, there is a constant c sucht that $\varphi_2 = \varphi_1 + c$ on W. Take $\varphi = \varphi_1$ on $f_1(U_1)$ and $\varphi = \varphi_2 - c$ on $f_2(U_2)$, then φ is a C^∞-function on $\mathbb{S}^2$ with $\omega = d\varphi$.
Use part 1 to conclude.